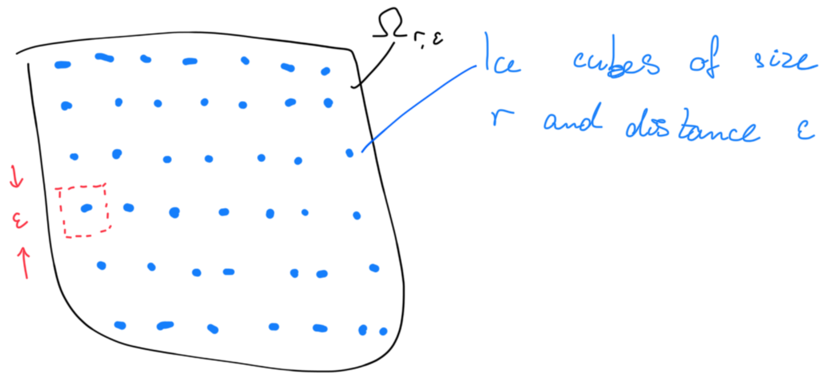
## Perforated domains:

Motivation: Crushed ice:




Temperature modelled by

$$\left.\begin{aligned} \partial_t u_{r,\varepsilon} - \Delta u_{r,\varepsilon} &= f \quad \text{in } \Omega_{r,\varepsilon} \\ u_{r,\varepsilon} &= 0 \quad \text{on } \partial\Omega_{r,\varepsilon} \end{aligned}\right\}$$

Heuristically:

- If $\varepsilon$ fixed, $r \to 0$, then in the limit

$$\partial_t u - \Delta u = f \quad \text{on } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

- If $r \sim \varepsilon$ as $\varepsilon \to 0$, then total mass of ice remains constant, while surface of ice grows unboundedly

  Mass: $m \approx \frac{1}{\varepsilon^N} \cdot r^N \sim \text{const.}$

  Surface: $\sigma \approx \frac{1}{\varepsilon^N} \cdot r^{N-1} \sim \frac{1}{\varepsilon}$

  $\rightsquigarrow \quad u_{r,\varepsilon} \longrightarrow 0 \quad \text{as} \quad r \sim \varepsilon \to 0.$

$\rightsquigarrow$ Question: What about intermediate scalings?

Let $\Omega \subset \mathbb{R}^N$ open, bounded, let $T_i^\varepsilon \subset \mathbb{R}^N$ closed for $1 \leq i \leq n(\varepsilon)$.

Define
$$\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^{n(\varepsilon)} T_\varepsilon^i$$

and for $f \in L^2(\Omega)$ consider

$$\left.\begin{aligned} -\Delta u_\varepsilon &= f \quad \text{in } \Omega_\varepsilon \\ u_\varepsilon &\in H_0^1(\Omega_\varepsilon) \end{aligned}\right\}$$

Weak formulation: Find $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$ s.t.

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla v \, dx = \int_{\Omega_\varepsilon} f v \, dx \qquad \forall v \in H_0^1(\Omega_\varepsilon).$$

Denote $\tilde{u}_\varepsilon := \begin{cases} u_\varepsilon & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \bigcup_i T_\varepsilon^i \end{cases}$. Then

$$\|\tilde{u}_\varepsilon\|_{H^1(\Omega)}^2 \overset{\text{Poincaré}}{\leq} C \|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)}^2 = C \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 = C \int_{\Omega_\varepsilon} f u_\varepsilon \, dx$$
$$\leq C \int_\Omega f \tilde{u}_\varepsilon \, dx$$
$$\leq C \|f\|_{L^2(\Omega)} \|\tilde{u}_\varepsilon\|_{L^2(\Omega)}$$

$\Longrightarrow$ $(\tilde{u}_\varepsilon)$ bounded in $H^1(\Omega)$

$\Longrightarrow$ conv. subsequence $\tilde{u}_\varepsilon \rightharpoonup u_0$ in $H^1(\Omega)$

$\rightsquigarrow$ $\tilde{u}_\varepsilon \longrightarrow u_0$ in $L^2(\Omega)$.

$\rightsquigarrow$ Question: Can $u_0$ be identified?

$$\int_\Omega \nabla \tilde{u}_\varepsilon \cdot \nabla w_\varepsilon \, \varphi \, dx + \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi \, w_\varepsilon \, dx = \int_\Omega f \, w_\varepsilon \, \varphi \, dx$$

$$\begin{array}{ccc} \downarrow & \downarrow{\scriptstyle (H3)} & \downarrow{\scriptstyle (H3)} \\ ? & \int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx & \int_\Omega f \varphi \, dx \end{array}$$

First term:

$$\int_\Omega \nabla \tilde{u}_\varepsilon \cdot \nabla w_\varepsilon \, \varphi \, dx = \int_\Omega \nabla w_\varepsilon \cdot \nabla (\tilde{u}_\varepsilon \varphi) \, dx - \underbrace{\int_\Omega \nabla w_\varepsilon \cdot \nabla \varphi \, u_\varepsilon \, dx}_{\downarrow \; 0}$$

$$= \langle -\Delta w_\varepsilon , \tilde{u}_\varepsilon \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + o(1)$$

$$\xrightarrow{(H5)} \langle \mu , u_0 \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}$$

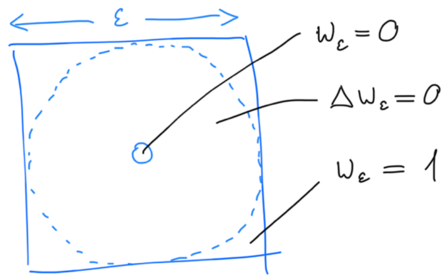
$\square$

<u>Example:</u>

$i \in \varepsilon \cdot \mathbb{Z}^N \subset \mathbb{R}^N$, $T^i_\varepsilon = \overline{B_{r_\varepsilon}(i)}$ closed balls around $i$,

$r_\varepsilon \ll \varepsilon$. Define $w_\varepsilon$ by

$$\begin{cases} w_\varepsilon = 0 & \text{in } T^i_\varepsilon \\ \Delta w_\varepsilon = 0 & \text{in } B_\varepsilon(i) \setminus T^i_\varepsilon \\ w_\varepsilon = 1 & \text{in corners} \\ w_\varepsilon & \text{continuous} \end{cases}$$




In polar coordinates:

$$w_\varepsilon(r) = \frac{r_\varepsilon^{2-N} - r^{2-N}}{r_\varepsilon^{2-N} - \varepsilon^{2-N}} \qquad (N \geq 3)$$

Define $\mu$ by

$$\mu = \frac{|\partial B_1(0)| \, (N-2)}{2^N} \qquad (\mu \text{ is just a number!})$$

**Theorem:** If we choose $r_\varepsilon = \varepsilon^{\overline{N-2}}$, then $w_\varepsilon$ and $\mu$ satisfy (H1) - (H5)

**Proof:**

Explicit computation:

$$\int_{\varepsilon\square} |\nabla w_\varepsilon|^2 dx = \frac{|\partial B_1(0)|(N-2)}{r_\varepsilon^{2-N} - \varepsilon^{2-N}}$$

$$\Rightarrow \sum_i \int_{\varepsilon\square} |\nabla w_\varepsilon|^2 dx \approx \varepsilon^{-N} \frac{|\partial B_1(0)|(N-2)}{r_\varepsilon^{2-N} - \varepsilon^{2-N}}$$

$\rightsquigarrow$ Need $\varepsilon^N r_\varepsilon^{2-N} \sim$ const.

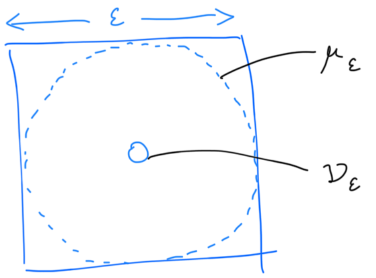
$\rightsquigarrow$ $r_\varepsilon = \varepsilon^{\frac{N}{N-2}}$.

Then $w_\varepsilon$ bounded in (H1) $\Rightarrow$ $w_\varepsilon \rightharpoonup w$.

Proof that $w = 1$: technical.

Proof of (H5):

By definition of $w_\varepsilon$: $-\Delta w_\varepsilon = \mu_\varepsilon - \nu_\varepsilon$, where




and $\langle \nu_\varepsilon, \varphi v_\varepsilon \rangle = 0$ $\forall v \in H_0^1(\Omega_\varepsilon)$.

$\Rightarrow$ Need only to compute limit of $\langle \mu_\varepsilon, \varphi v_\varepsilon \rangle$

$\mu_\varepsilon$ acts by averaging over $\partial B_\varepsilon(i)$ and summing over $i$.

$\rightsquigarrow$ For small $\varepsilon$: $\langle \mu_\varepsilon, g \rangle$ is like Riemann sum.

$\rightsquigarrow$ $\langle \mu_\varepsilon, g \rangle \longrightarrow (\text{const.}) \int_\Omega g \, dx$ $\square$